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On theory of end resonance in semi-infinite solid elastic cylinders

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Starting from Mindlin and Herrmann equations exact theoretical solutions for the elastic waves in terminated solid cylinders have been obtained by an integral transform method. The result predicts end resonance and the predicted resonant frequency tallies with the experimental value remarkably. The theory in addition gives two other resonant frequencies for the end vibration which might be real or complex in nature depending upon the value of the Poisson's ratio of the material of the cylinder.

1. INTRODUCTION

Exact theoretical solutions for elastic wave propagation in terminated wave-guides are not reported in the literature, fundamentally because of the well-known difficulty in solving the wave equations in media bounded by two intersecting surfaces. Thus complete and rigorous mathematical solutions for elastic waves near the terminated ends of semi-infinite solid cylinders are not known although some interesting observations were made in some related experiments.

Bishop (1953) suggested the possible existence of a wave system confined to the ends of a rod to account for the unbalanced stresses near such boundaries. He apparantly failed to forsee any resonance effect at the end of the rod. The most interesting and intuitive observations were made in the almost classic experiment of Oliver (1957) who showed the existence of a high Q resonance of the end zones of a cylinder. McNiven (1961) started from a three mode theory and succeeded in satisfying the boundary conditions at the end surface of a cylinder which is excited by a low frequency pulse travelling in the lowest longitudinal mode, the frequency of propagation being assumed to be lower than the lowest cut-off frequency of the next higher mode. The result shows the presence of an end vibration but the quantitative agreement with experiments is poor. The defect in the above theory is that one component of the end stresses has

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only been averaged out instead of satisfying the condition of vanishing stress throughout the surface. Thus a considerable magnitude of the stress component might easily be present over the whole surface without disturbing the assumed boundary condition.

Zemanek (1962), on the other hand, found an approximate solution for the mechanism of reflection of a pulse propagating in the lowest longitudinal mode presuming an undistorted reflected signal together with a number of complex modes assumed to be generated at the end surface on incidence of the primary pulse. He chose some equispaced concentric circles over the end face and succeeded in satisfying the boundary conditions over these circles. The result gives end resonance which tallies with the experimental results. The shortcoming in the above theory is that the entire annular spaces in between a few stress free circles over the end face are left free from imposition of any boundary condition. Moreover, the method of calculation of Zemanek consists of elaborate numerical computations. Any further result one desires to have can only be obtained after extensive computer work which in return does not give any extra insight into the mechanism involved.

The present work describes an integral transform technique of solving an end problem starting from the wellknown Mindlin & Herrmann (1951) equation. The result shows end resonance with some accuracy apart from pointing out a few more interesting aspects.

2. Theory

The coordinate system chosen for the present study is cylindrical polar (r, θ, z) with its origin at the centre of the free end surface of the semi-infinite solid cylinder, with its z axis coincident with the symmetry axis of the cylinder, the cylinder extending towards $+z$ direction and the free cylindrical lateral surface of the cylinder being designated by $r = a$. The theory derived by Mindlin & Herrmann (1951) for the propagation of elastic waves in a solid cylinder is contained in the following differential equations

$$\left.\begin{aligned} a^2K^2\mu \frac{\partial^2 u(z,t)}{\partial z^2} - 8K_1^2(\lambda+\mu)u(z,t) - 4aK_1^2\lambda \frac{\partial w(z,t)}{\partial z} &= \rho a^2 \frac{\partial^2 u(z,t)}{\partial t^2} \\ 2a\lambda \frac{\partial u(z,t)}{\partial z} + a^2(\lambda+2\mu)\frac{\partial^2 w(z,t)}{\partial z^2} &= \rho a^2 \frac{\partial^2 w(z,t)}{\partial t^2} \end{aligned}\right\} \quad \ldots (1)$$

where λ and μ are the two Lame constants in their usual sense and ρ is the density of the material of the rod, $u(z, t)$ and $w(z, t)$ are the radial and axial components of displacement function, and K and K_1 are the well known correction constants introduced by Mindlin & Herrmann (1951). The radial parts of the equations of motion have been eliminated by Mindlin & Herrmann (1951) following a careful satisfies the boundary conditions at the lateral surface $r = a$ of the cylinder. process which

Assuming harmonic variation $\exp(i\omega t)$ the above differential equations of motion become

$$a^2K^2\mu \frac{\mathrm{d}^2u(z)}{\mathrm{d}z^2}+[-8K_1^2(\lambda+\mu)+\rho a^2\omega^2]u(z)-4aK_1^2\lambda\frac{\mathrm{d}w(z)}{\mathrm{d}z} = 0, \quad \ldots \quad (2a)$$

$$2a\lambda\frac{\mathrm{d}u(z)}{\mathrm{d}z}+a^2(\lambda+2\mu)\frac{\mathrm{d}^2w(z)}{\mathrm{d}z^2}+\rho a^2\omega^2w(z) = 0. \quad \ldots \quad (2b)$$

The boundary conditions at the end surface for the present problem are

$$[T_{zz}]_{z=0} = 0$$

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where T_{zz} and T_{rz} are the stress components in their conventional sense. These conditions when expressed in terms of the displacement components, utilizing the derivation process of Mindlin & Herrmann (1951), become

$$\left[u(z)+\frac{a(\lambda+2\mu)}{2\lambda}\frac{\mathrm{d}w(z)}{\mathrm{d}z}\right]_{z=0} = 0, \quad \ldots \quad (3a)$$

$$\left[\frac{\mathrm{d}u(z)}{\mathrm{d}z}\right]_{z=0} = 0. \quad \ldots \quad (3b)$$

To solve the above differential equations subject to the boundary conditions stated above let us multiply eq. (2*a*) by $\cos pz$ and eq. (2*b*) by $\sin pz$ and integrate them from 0 to ∞ with respect to z and utilizing boundary conditions (3*b*) one obtains

$$\{-a^2K^2\mu p^2-8K_1^2(\lambda+\mu)+\rho a^2\omega^2\}u_c(p)-4aK_1^2\lambda p w_s(p) = -4aK_1^2\lambda w(0), \quad \ldots \quad (4a)$$

$$-2a\lambda p u_c(p)+\{-a^2(\lambda+2\mu)p^2+\rho a^2\omega^2\}w_s(p) = -a^2(\lambda+2\mu)pw(0), \quad \ldots \quad (4b)$$

where $u_c(p)$ and $w_s(p)$ are the Fourier cosine transform of $u(z)$ and Fourier sine transform of $w(z)$ respectively.

Solving eqs. (4*a*) and (4*b*) we obtain

$$w_s(p) = \frac{(a_1p^3+a_2p)w(0)}{b_1p^4+b_2p^2+b_3}, \quad \ldots \quad (5a)$$

$$u_c(p) = \frac{(a_3p^2+a_4)w(0)}{b_1p^4+b_2p^2+b_3}, \quad \ldots \quad (5b)$$

where

$$
\begin{aligned}
a_1 &= a^4K^2(\lambda+2\mu)\mu \\
a_2 &= -8a^2{K_1}^2\lambda^2+8{K_1}^2a^2(\lambda+\mu)(\lambda+2\mu)-\rho a^4\omega^2(\lambda+2\mu) \\
a_3 &= 0 \\
a_4 &= -4\rho a^3\omega^2{K_1}^2\lambda \\
b_1 &= a^4K^2\mu(\lambda+2\mu) \\
b_2 &= -\rho a^4\omega^2K^2\mu+8{K_1}^2a^2(\lambda+\mu)(\lambda+2\mu)-\rho a^4\omega^2(\lambda+2\mu)-8{K_1}^2a^2\lambda^2 \\
b_3 &= \rho^2a^4\omega^4-8\rho a^2\omega^2{K_1}^2(\lambda+\mu).
\end{aligned} \tag{6}
$$

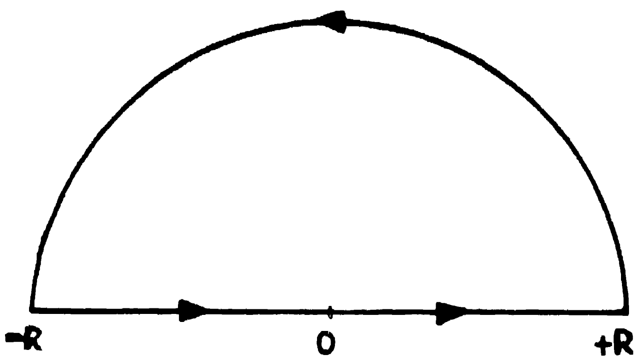

Fig. 1. The contour for the evaluation of integrals involving inversions of the expressions in eqs. (5*a*) and (5*b*).

In order to obtain the inversions of eqs. (5*a*) and (5*b*) the contour of integration chosen is shown in figure 1, two of the four poles being enclosed by the contour. Proceeding in the usual manner and after some calculations one obtains the required inversions as

$$
w(z,t) = \frac{w(0)\operatorname{cosec} 2\theta}{b_1}\left[\exp(-\alpha z\cos\theta)\left\{a_1\sin(2\theta-\alpha z\sin\theta) + \frac{a_2}{\alpha^2}\sin\alpha z\sin\theta\right\}\right]\exp(i\omega t), \quad \ldots \quad (7a)
$$

$$
u(z,t) = \frac{w(0)\operatorname{cosec} 2\theta}{b_1}\,\frac{a_4}{\alpha^3}\,[\exp(-\alpha z\cos\theta)\{\sin(\theta+\alpha z\sin\theta)\}]\exp(i\omega t), \quad \ldots \quad (7b)
$$

where

$$
\cos 2\theta = b_2/2(b_1b_3)^{1/2}
$$

$$
\alpha = (b_3/b_1)^{1/4}.
$$

The above form of solution is not valid when $b_3 = 0$. In this case which corresponds to a particular frequency for the system separate inversions yield the expressions

$$w(z, t) = \left[\left(\frac{a_1}{b_1} - \frac{a_2 b_1}{b_2^2}\right) \cos\left(\frac{b_2 z}{b_1}\right) + \frac{a_2 b_1}{b_2^2}\right] w(0)\exp(i\omega t). \quad \ldots \quad (7c)$$

$$u(z, t) = \left[-\frac{a_4 b_1^2}{b_2^3} \sin\left(\frac{b_2 z}{b_1}\right) + \frac{a_4 b_1}{b_2^2} z\right] w(0)\exp(i\omega t). \quad \ldots \quad (7d)$$

The boundary condition (3*a*) will now be utilized and eqs. (7*a*) and (7*b*) give the frequency equation of the system as

$$\frac{a_4}{a_1} + K_3 \alpha^4 \left[-1 - 2\cos 2\theta + \frac{a_2}{a_1 \alpha^2}\right] = 0, \quad \ldots \quad (8)$$

where $K_3 = a(\lambda + 2\mu)/2\lambda$.

Eqs. (7*c*) and (7*d*), on the other hand, automatically satisfy the boundary condition (3*a*). Thus $b_3 = 0$ gives a resonance condition at a frequency

$$\frac{\rho a^2 \omega^2}{\mu} = \frac{8K_1^2}{1 - 2\sigma}, \quad \ldots \quad (9)$$

where σ is the Poisson's ratio of the material of the cylinder.

On further simplifications eq. (8) gives

$$(\bar{\omega}^2 - \chi_1^2)^{\frac{1}{2}}[(\bar{\omega}^2 - \chi_1^2)^{\frac{1}{2}}\{K\bar{\omega} - \chi_3(\bar{\omega}^2 - \chi_1^2\}^{\frac{1}{2}} - \chi_2^2] = 0, \quad \ldots \quad (10)$$

where

$$\chi_1^2 = 8K_1^2 \frac{\lambda + \mu}{\mu}, \qquad \chi_3^2 = (\lambda + 2\mu)/\mu,$$

$$\chi_2^2 = 8K_1^2 \lambda^2 / \{(\lambda + 2\mu)\mu^3\}^{\frac{1}{2}}, \qquad \bar{\omega}^2 = \rho a^2 \omega^2 / \mu.$$

Eq. (10) above suggests two separate solutions. The solution corresponding to the former factor has already been discussed in eq. (9). The other solution gives the other resonance frequencies as

$$\omega_2^2 = \frac{16K_1^2\sigma + 4K_1^2(4 - K^2) \pm [\{16K_1^2\sigma + 4K_1^2(4 - K^2)\}^2 - 128K_1^4\{2\sigma(K^2 - 1) - K^2 + 2\}]^{\frac{1}{2}}}{2\sigma(K^2 - 1) + 2 - K^2}. \quad \ldots \quad (11)$$

3. Results and Discussion

Figure 2 depicts a plot of ω_1/K_1 against Poisson's ratio of the material of the cylinder. In order to obtain the actual resonant frequencies from this plot corresponding to the cylinders of different diameters and having different material constants one has to bear in mind that K and K_1 are also functions of the Poisson's ratio. The method of calculation of these correction factors for any value of Poisson's ratio have been described by Mindlin & Herrmann (1951).

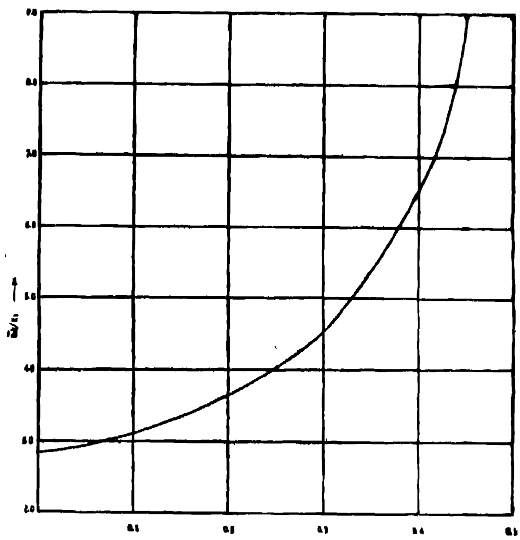

Fig. 2. The variation of ω_1/K_1 with the Poisson's ratio of the material of the cylinder.

For further numerical evaluations let us take $\sigma = 0{\cdot}325$. The above mentioned method of Mindlin & Herrmann (1951) gives the values of the correction constants as $K = 0{\cdot}932$ and $K_1 = 0{\cdot}691$. For an experimental verification of the theory presented here let us take the values of the Oliver's experiment.

$$\text{The radius of the cylinder} = 0{\cdot}50 \text{ inches}$$
$$(\mu/\rho)^{\frac{1}{2}} = 1{\cdot}05 \times 10^4 \text{ feet per second.}$$

These values when substituted in the result obtained in the present paper predicts an end resonance period 7.70μ seconds as against the experimentally obtained value of Oliver (1957) 8.4μ seconds. The accuracy is fairly good. A comparative study of the various experimental and theoretical results is given in table 1 for $\sigma \approx 0.33$.

Table 1 Experimental and calculated values of $\bar{\omega}_1$

Observer	Theoretical $\bar{\omega}_1$	Experimental $\bar{\omega}_1$
Oliver	3.524	3.000
McNiven	2.590	——
Zomanek	3.006	3.026
Present Theory	3.271	——

The other roots for the present case as calculated from eq. (11) are complex in nature

$$\bar{\omega}_2 = 8{\cdot}305+0{\cdot}160i \quad \text{and} \quad -0{\cdot}216-6{\cdot}157i.$$

where $$i = (-1)^{\frac{1}{2}}.$$

The author wants to make a few more comments on the multiple nature of the roots of $\bar{\omega}$. Though in the present example only one root is real, this is not a general happening for all the values of the Poisson's ratio. Thus for the limiting case $\sigma = 0$ similar proceedure of calculation gives

$$K = 0{\cdot}858$$

$$K_1 = 0{\cdot}723$$

$$\bar{\omega}_1 = 1{\cdot}479$$

$$\bar{\omega}_2 = 2{\cdot}045 \quad \text{and} \quad 2{\cdot}602.$$

In this case ω has three real roots predicting three real resonant frequencies.

Further it becomes evident here that a similar theory developed from a more general theory for terminated cylinders is expected to give set of such frequencies. The hints for the existence of a number of such resonances have already been obtained by Oliver (1957) in his experiment.

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